

Optimization and Decision Making

2019

Time allowed: 2 hours

*Full marks may be obtained for correct answers to
THREE of the FOUR questions.
All necessary working must be shown.*

1. A bus company is considering starting a new bus service in a town. The town is a tourist destination, and the main business will be over the Summer, but the company can test the likely demand in the Spring by operating a Limited service, which will give some idea of what will happen over the Summer (it will not start a service in the Summer without running this Limited Spring service). In the Spring, passenger numbers can be designated either as High (Spring) or Low (Spring). The company estimates that there is a 0.6 chance of High (Spring) passenger numbers. If passenger numbers in the Spring are low, it will cost the company £120000, and they will abandon plans of operating any service in the Summer. If they are high, they will then have enough confidence to run a service in the Summer. This service can be Limited or Full. The profits achieved would depend upon the nature and level of passenger numbers, which we categorize as either High (Summer) or Low (Summer).

If the company opted for a Full service and numbers were high, a £300000 profit would be obtained. If the numbers were low in this case, however, the company would make a loss of £240000.

If the company decided to go for the Limited service, High passenger numbers would generate a profit of £180000 and Low numbers would generate a profit of £60000.

The company estimates that there is a 0.75 chance of High (Summer) passenger numbers in the Summer if there were High (Spring) numbers in the Spring.

- (a) Construct a decision tree to represent this decision problem. [15]
- (b) Assuming that the company's objective is to maximize its expected profit, determine the policy that it should adopt. [5]
- (c) The company's estimated probability of High (Spring) numbers may not be correct. Assuming that all other problem elements remain the same, determine how low this probability would have to be before the option of not running any service should be chosen. [5]
- (d) Before the final decision is taken a new Finance Director joins the company, who insists that a utility theory approach is taken to the problem, and who gives the utilities shown below for the sums of money involved in the decision. What implications does this have for the policy that you found from b) and why? [5]

Value (£ million)	-240000	-120000	0	60000	180000	300000
Utility	0	0.4	0.7	0.8	0.95	1

2. Containers arrive at four different ports, and are distributed to warehouses in three different cities.

Ports A, B, C and D have received 4, 3, 7 and 6 containers respectively. Warehouses 1, 2 and 3 require the contents of 5, 5 and 10 containers respectively.

The transportation costs (in pounds) of supplying one container from a given port to a given city are given in the following table. The aim is to minimise the total cost of transporting the containers.

	1	2	3
A	160	150	190
B	150	140	220
C	100	170	180
D	120	210	210

- (a) Formulate, but do not solve, the problem as a linear program. [7]
- (b) Now write the problem in the form of a transportation array and hence find the optimal transportation solution. [16]
- (c) How would the solution to the above problem change if the cost of transportation from port D to depot 1 rose to 140? [7]
3. Consider the primal linear program below.

Maximise $x_1 + 2x_2 + 3x_3$,

subject to

$$x_1 + x_2 + 4x_3 \leq 12,$$

$$x_1 + 3x_2 + x_3 \leq 9,$$

$$x_1, x_2, x_3 \geq 0.$$

- (a) Construct the dual problem and solve it graphically. [12]
- (b) Without using simplex tableau, determine a solution to the primal problem, stating clearly any results that you use. [10]
- (c) Now suppose that the first constraint in the primal problem $x_1 + x_2 + 4x_3 \leq 12$ is replaced by $x_1 + x_2 + 4x_3 \leq 2$. Find the solution to the dual program. Explain what this means for the solution to the primal program. [8]

4. Suppose that you have to choose an optimal portfolio from a list of n stocks. Stock i has expected revenue rate μ_i with variance σ_i^2 for $i = 1, \dots, n$, and the covariance of the revenues of stocks i and j is given by σ_{ij} for $i \neq j, i, j = 1, \dots, n$. The proportion of stock i in the portfolio is denoted by w_i .

- (a) Show that the expected revenue from the portfolio is $\sum_{i=1}^n w_i \mu_i$, and find an expression for the variance of the portfolio revenue. [8]
- (b) Formulate the general n stock problem using Lagrange multipliers, and hence write down a set of equations for the optimal value of the w_i s. Hence use this set of equations to show that the optimal solution for the 3 stock problem is given by the following equations:

$$\begin{aligned} & [\mu_2(\sigma_1^2 - \sigma_{13}) + \mu_1(\sigma_{13} - \sigma_{12}) + \mu_3(\sigma_{12} - \sigma_1^2)]w_1 + \\ & [\mu_3(\sigma_2^2 - \sigma_{12}) + \mu_2(\sigma_{12} - \sigma_{23}) + \mu_1(\sigma_{23} - \sigma_2^2)]w_2 + \\ & [\mu_1(\sigma_3^2 - \sigma_{23}) + \mu_3(\sigma_{23} - \sigma_{13}) + \mu_2(\sigma_{13} - \sigma_3^2)]w_3 = 0, \\ & w_2 = \frac{(\mu_P - \mu_3) - w_1(\mu_1 - \mu_3)}{\mu_2 - \mu_3}, \\ & w_3 = \frac{(\mu_2 - \mu_P) + w_1(\mu_1 - \mu_2)}{\mu_2 - \mu_3}. \end{aligned}$$

[10]

- (c) Now consider a problem with three stocks where the means, variances and covariances are as follows:
 $\mu_1 = 0.06, \mu_2 = 0.07, \mu_3 = 0.10, \sigma_1^2 = 0.2, \sigma_2^2 = 0.2, \sigma_3^2 = 0.5, \sigma_{12} = 0.1, \sigma_{13} = -0.1$ and $\sigma_{23} = 0.1$.
 Find the optimal portfolio (i.e. the one with the minimum variance) if the targeted expected rate of return is 0.07. [8]
- (d) Suppose that the targeted expected rate of return now changes to 0.08. What is the optimal portfolio now? [4]

Internal Examiner: Professor M. Broom
 External Examiner: Professor D. Gounopoulos